Int. J. Solids Structures Vol. 25, No. 12, pp. 1467–1482, 1989
Printed in Great Britain.

0020-7683/89 \$3.00 + .00

THERMOELASTIC STRESS ANALYSIS OF MODERATELY THICK INHOMOGENEOUS AND LAMINATED PLATES

T. G. ROGERS and A. J. M. SPENCER
Department of Theoretical Mechanics, University of Nottingham, Nottingham NG7 2RD, U.K.

(*Received* 21 *November* 1988; *in revised form* 3 *April* 1989)

Abstract—We extend the plane stress theory of Michell (1900, *Proc. Lond. Math. Soc.* **31**, 100–124) for a moderately thick homogeneous elastic plate, and that of Kaprielian *et al.* (1988, *Phil Trans. R. Soc. Lond.* **A324**, 565–594) for a laminated plate, to include stretching and bending solutions for an inhomogeneous thermoelastic plate. The inhomogeneities, both in the elastic properties and thermal expansion coefficients, can vary arbitrarily through the thickness of the plate, though for simplicity the analysis is restricted to plates with geometric and material properties symmetric with respect to the mid-plane. The deformation is produced by a temperature field which can also vary arbitrarily through the thickness.

The solutions are expressed in terms of the solution of the approximate, two-dimensional, thin-plate equations governing an "equivalent" homogeneous plate, and are exact solutions of the full equations of three-dimensional thermoelasticity. By considering laminated plates to be a special case of inhomogeneous plates, we derive an "exact" laminate theory for plates consisting of different, homogeneous and isotropic layers which are perfectly bonded to each neighbouring layer.

1. INTRODUCTION

In a recent paper, Kaprielian *et al.* (1988) established a theory which gives exact solutions of the three-dimensional elasticity equations for a large class of problems involving the stretching and bending of laminated elastic plates under the action of edge forces and couples. The theory is based on a generalization of Michell's exact plane stress theory (Michell, 1900) for a homogeneous isotropic elastic plate, and enables all the interface traction and displacement continuity conditions to be satisfied, as well as satisfying the zero traction condition on the lateral surfaces. A feature of the theory is that the solutions are expressed in terms of solutions of the approximate two-dimensional thin-plate theory (Timoshenko and Woinowsky-Krieger, 1959) for an "equivalent" plate of the same overall geometry; once these "equivalent" solutions are obtained then the corresponding exact three-dimensional solution may be derived by simple substitution. Furthermore, although the three-dimensional solution is expressed in analytical form it is not necessary that the thin-plate solution also be obtained analytically. Thus existing numerical codes for solving the thin-plate equations can be extended to give three-dimensional solutions, to within an accuracy determined by that of the numerical procedure (and not by the accuracy or otherwise of the approximate thin-plate theory).

In a subsequent development we have generalized Michell's solution not only to laminates but also to plates with arbitrary inhomogeneity through the thickness. In two companion papers in a volume dedicated to I. N. Sneddon, we have presented the stretching (Spencer, 1989) and bending (Rogers, 1989) solutions equivalent to those presented in Kaprielian *et al.* (1988) for laminated plates, together with the solution (Rogers, 1989) for bending under the action of a uniform pressure applied to one of the faces. Again the method was to formulate the solution in terms of the solution of the equivalent thin-plate equations, and again it was found that the corresponding exact solution of the full equations of three-dimensional elasticity is obtained by simple substitutions. The analyses also confirmed the theory in Kaprielian *et al.* (1988) as the important special case of "piecewise-constant" inhomogeneity. For convenience, the inhomogeneous plates were assumed to be symmetric in the sense that the mid-plane is a plane of reflectional symmetry for the plate geometry; the general case is the subject of a future paper.

In this paper we demonstrate these methods, and make a modest extension to the theory, by including the effects of specified temperature fields being imposed on such plates. The inhomogeneity now applies to the thermal properties as well as to the elastic moduli. For convenience we again restrict attention to symmetric plates, and we consider only temperature fields that are spatially dependent on the through-thickness coordinate. As in the previous papers, this leads to the uncoupling of the stretching and bending deformation modes.

The analysis is confined to inhomogeneous isotropic plates. Whilst the effect of thermal stresses in any laminate has an obvious practical importance, the application of greatest current interest is probably to fibre-reinforced laminates in which each layer is not isotropic but transversely isotropic or orthotropic (Christensen, 1979). Nonetheless, we believe a three-dimensional theory of isotropic plates is important for at least three reasons, as described in Kaprielian *et al.* (1988): a new class of exact solutions in three-dimensional elasticity has an intrinsic merit in itself, the analysis can give a strong indication of the manner in which we should proceed with the analysis of anisotropic laminae (Kaprielian, 1985), and the solutions, being exact, can be used as precise tests of numerical procedures (or software packages) for the stress analysis of actual laminates.

In the following section (Section 2), we present the full governing equations and specified boundary conditions for the problem. The only approximation allowed is that the edge boundary conditions are satisfied only in an average sense, rather than point by point. Thus the solution is valid everywhere except in edge boundary layers whose width is of the order of the plate thickness provided† the three-dimensional edge data are properly incorporated in the boundary conditions of the plate theory using the techniques of Gregory and Wan (1984, 1985).

As in Kaprielian *et al.* (1988), Spencer (1989) and Rogers (1989), we seek to express the solution in terms of that given by the simpler equations of thin-plate theory, and these are briefly reviewed in Section 3. A symmetric temperature field will produce a stretching deformation, whilst an anti-symmetric field produces bending. Since any temperature field can be expressed as the sum of a symmetric and an anti-symmetric field, the general solution decomposes into two independent solutions which we treat separately in Sections 4 and 5. Finally, in Section 6, we show the implications of these solutions for the special case of laminated plates. Like the theory described in Kaprielian *et al.* (1988), the present theory reduces to solving simple recurrence relations for constants describing the solution in each individual layer. Unlike that paper, we formulate the relations so that these constants are directly related to the more important *interlaminar* values of the displacement and stress components.

2. INHOMOGENEOUS THERMOELASTIC PLATES

We consider a plate of uniform thickness $2h$, whose mid-plane coincides with the plane $z = 0$ of a rectangular Cartesian system of axes with coordinates x, y, z. All vector and tensor components are referred to this system. The components of displacement are denoted by u, v, w and the components of the symmetric stress tensor $\boldsymbol{\sigma}$ by

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}. \tag{1}$$

The material is isotropic and linearly thermoelastic, with its elastic and thermal properties dependent on z, but independent of x and y. Hence the constitutive equations can be expressed as

† The authors are indebted to a referee for drawing this provision to their attention.

$$\begin{pmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\end{pmatrix} = \begin{pmatrix}\lambda+2\mu & \lambda & \lambda\\ \lambda & \lambda+2\mu & \lambda\\ \lambda & \lambda & \lambda+2\mu\end{pmatrix}\begin{pmatrix}u_{,x}\\ v_{,y}\\ w_{,z}\end{pmatrix} - \beta T\begin{pmatrix}1\\ 1\\ 1\end{pmatrix}, \tag{2}$$

and

$$\begin{pmatrix}\sigma_{yz}\\ \sigma_{zx}\\ \sigma_{xy}\end{pmatrix} = \mu\begin{pmatrix}v_{,z}+w_{,y}\\ w_{,x}+u_{,z}\\ u_{,y}+v_{,x}\end{pmatrix}, \tag{3}$$

where commas denote partial differentiation with respect to the suffix variables, and the Lamé elastic moduli λ, μ and the stress–temperature coefficient of linear thermal expansion β are functions of z. The temperature field T is taken to be independent of x and y, and quasi-static conditions are assumed so that the time t takes on the role of a parameter and is accordingly omitted from the analysis. Furthermore, for convenience we express T as the sum of its symmetric and anti-symmetric parts, with

$$T(z) = T_e(z) + T_o(z), \tag{4}$$

where

$$T_e(z) = \tfrac{1}{2}\{T(z)+T(-z)\}, \quad T_o(z) = \tfrac{1}{2}\{T(z)-T(-z)\}, \tag{5}$$

with T_e an even function of z and T_o an odd function:

$$T_e(z) = T_e(-z), \quad T_o(z) = -T_o(-z). \tag{6}$$

Since the plate is symmetric, λ, μ and β are all even functions of z:

$$\lambda(z) = \lambda(-z), \quad \mu(z) = \mu(-z), \quad \beta(z) = \beta(-z). \tag{7}$$

We also introduce for later use in the analysis the quantities

$$\eta = \lambda/(\lambda+2\mu), \quad \gamma = \beta/(\lambda+2\mu) \tag{8}$$

and

$$l = 2\mu(\eta+1), \quad \beta^* = 2\mu\gamma. \tag{9}$$

The equations of equilibrium, assuming body forces to be zero, are

$$\begin{aligned}\sigma_{xx,x}+\sigma_{xy,y}+\sigma_{xz,z} &= 0,\\ \sigma_{yx,x}+\sigma_{yy,y}+\sigma_{yz,z} &= 0,\\ \sigma_{zx,x}+\sigma_{zy,y}+\sigma_{zz,z} &= 0.\end{aligned} \tag{10}$$

The boundary condition of zero traction on the lateral surfaces $z = \pm h$ requires

$$\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0 \quad \text{on} \quad z = \pm h. \tag{11}$$

The through-thickness averages of u and v are defined by

$$\{\bar{u}(x,y), \bar{v}(x,y)\} = \frac{1}{2h}\int_{-h}^{h}\{u(x,y,z), v(x,y,z)\}\,\mathrm{d}z, \tag{12}$$

and the transverse displacement of the middle surface is denoted by

$$\bar{w}(x,y) = w(x,y,0). \tag{13}$$

Stress resultants are defined by

$$(N_{xx}, N_{yy}, N_{xy}, \ldots) = \int_{-h}^{h} (\sigma_{xx}, \sigma_{yy}, \sigma_{xy}, \ldots)\, dz \tag{14}$$

and the bending moments by

$$(M_{xx}, M_{yy}, M_{xy}, \ldots) = \int_{-h}^{h} (\sigma_{xx}, \sigma_{yy}, \sigma_{xy}, \ldots) z\, dz. \tag{15}$$

In terms of these quantities, the plate equilibrium equations take the integrated forms (Timoshenko and Woinowsky-Krieger, 1959):

$$N_{xx,x} + N_{xy,y} = 0, \quad N_{xy,x} + N_{yy,y} = 0, \tag{16}$$

and

$$M_{xx,xx} + 2M_{xy,xy} + M_{yy,yy} = 0, \tag{17}$$

where we have incorporated the boundary condition (11).

The formulation of the problem is completed by specifying appropriate conditions (Timoshenko and Woinowsky-Krieger, 1959) on the edge of the plate. Typical boundary conditions are that at the edge of a plate with outward unit normal $\mathbf{n} = (n_x, n_y, 0)$ we may specify one from each of the following pairs:

$$\bar{u}_n \text{ or } N_{nn}, \quad \bar{u}_s \text{ or } N_{ns}, \quad \bar{w} \text{ or } N_{nz} - \partial M_{ns}/\partial s, \quad \partial \bar{w}/\partial n \text{ or } M_{nn}, \tag{18}$$

where the suffices n and s denote components in the normal and tangential directions at the given point on the edge, so that, for example,

$$\bar{u}_n = n_x \bar{u} + n_y \bar{v}, \quad \partial/\partial s \equiv -n_y \partial/\partial x + n_x \partial/\partial y,$$
$$M_{ns} = (M_{yy} - M_{xx}) n_x n_y + M_{xy}(n_y^2 - n_x^2). \tag{19}$$

The first three pairs of (18) correspond to the specification of the mean edge displacement or the edge tractions, the fourth to specification of either the slope of the mid-surface or the bending moment applied to the edge of the plate.

3. CLASSICAL THIN-PLATE THEORY

In classical thin-plate theory it is assumed that plane sections that are initially normal to the middle surface remain plane and normal to the middle surface. Accordingly the in-plane displacement is approximated as

$$u(x,y,z) \simeq \bar{u}(x,y) - z\bar{w}(x,y)_{,x},$$
$$v(x,y,z) \simeq \bar{v}(x,y) - z\bar{w}(x,y)_{,y}. \tag{20}$$

It is also assumed that plane stress conditions apply so that in particular

$$\sigma_{zz} = 0. \tag{21}$$

Hence, from (2) and (8),

$$w_{,z} = -\eta(u_{,x}+v_{,y})+\gamma T. \tag{22}$$

Substituting (20) and (22) into the constitutive equations then gives, in an obvious notation,

$$\begin{pmatrix}\sigma_{xx}\\ \sigma_{yy}\end{pmatrix} \simeq \begin{pmatrix}\bar{\sigma}_{xx}\\ \bar{\sigma}_{yy}\end{pmatrix} = \begin{pmatrix} l & l-2\mu \\ l-2\mu & l\end{pmatrix}\begin{pmatrix}\bar{u}_{,x}-z\bar{w}_{,xx}\\ \bar{v}_{,y}-z\bar{w}_{,yy}\end{pmatrix} - T^*\begin{pmatrix}1\\ 1\end{pmatrix}, \tag{23}$$

where

$$T^* = \beta^* T = 2\mu\gamma T. \tag{24}$$

Hence we obtain

$$\begin{aligned} N_{xx} \simeq \bar{N}_{xx} &= 2h\{\hat{l}\bar{u}_{,x}+(\hat{l}-2\hat{\mu})\bar{v}_{,y}-\hat{T}^*_e\},\\ N_{yy} \simeq \bar{N}_{yy} &= 2h\{(\hat{l}-2\hat{\mu})\bar{u}_{,x}+\hat{l}\bar{v}_{,y}-\hat{T}^*_e\},\\ N_{xy} \simeq \bar{N}_{xy} &= 2h\hat{\mu}(\bar{u}_{,y}+\bar{v}_{,x}), \end{aligned} \tag{25}$$

where

$$(\hat{l}, \hat{\mu}, \hat{T}^*_e) = \frac{1}{2h}\int_{-h}^{h} (l, \mu, T^*_e)\,\mathrm{d}z = \frac{1}{h}\int_0^h (l, \mu, T^*_e)\,\mathrm{d}z. \tag{26}$$

Thus, $\hat{l}$, $\hat{\mu}$ and $\hat{T}^*_e$ represent average values of l, μ and T^*_e through the plate thickness; since T^*_o is an odd function of z, then $\hat{T}^*_o$ is zero. Here $\hat{l}$ and $\hat{\mu}$ are constants defined by the plate properties alone, whereas $\hat{T}^*_e$ is a constant which is also dependent on the particular temperature field that is applied. For a homogeneous plate $\hat{l} = l$, $\hat{\mu} = \mu$ and $\hat{T}^*_e = 2\mu\gamma\hat{T}_e$.

Substitution of (25) into (16) gives the same thin-plate equations as for isothermal elasticity (Timoshenko and Woinowsky-Krieger, 1959):

$$\hat{l}\bar{\Delta}_{,x}-\hat{\mu}\bar{\Omega}_{,y} = 0, \quad \hat{l}\bar{\Delta}_{,y}+\hat{\mu}\bar{\Omega}_{,x} = 0, \tag{27}$$

where, as in Kaprielian *et al.* (1988) and Spencer (1989), it is found convenient to introduce

$$\bar{\Delta} = \bar{u}_{,x}+\bar{v}_{,y}, \quad \bar{\Omega} = \bar{v}_{,x}-\bar{u}_{,y}. \tag{28}$$

It follows immediately from (27) that

$$\nabla^2\bar{\Delta} = 0, \quad \nabla^2\bar{\Omega} = 0, \tag{29}$$

where ∇^2 is the two-dimensional Laplacian operator

$$\nabla^2 \equiv \partial^2/\partial x^2+\partial^2/\partial y^2. \tag{30}$$

Similarly, substitution of (23) into (15) yields

$$\begin{aligned} M_{xx} \simeq \bar{M}_{xx} &= -\tfrac{2}{3}h^3\{\tilde{l}\bar{w}_{,xx}+(\tilde{l}-2\tilde{\mu})\bar{w}_{,yy}-\tilde{T}^*_o\},\\ M_{yy} \simeq \bar{M}_{yy} &= -\tfrac{2}{3}h^3\{(\tilde{l}-2\tilde{\mu})\bar{w}_{,xx}+\tilde{l}\bar{w}_{,yy}-\tilde{T}^*_o\},\\ M_{xy} \simeq \bar{M}_{xy} &= -\tfrac{4}{3}h^3\tilde{\mu}\bar{w}_{,xy}, \end{aligned} \tag{31}$$

where $\tilde{l}$, $\tilde{\mu}$ and $\tilde{T}^*_o$ are weighted averages defined by

$$(\bar{l}, \bar{\mu}) = \frac{3}{2h^3}\int_{-h}^{h} \{l(z), \mu(z)\} z^2 \, dz, \quad \tilde{T}_o^* = \frac{3}{2h^3}\int_{-h}^{h} T_o^*(z) z \, dz. \tag{32}$$

Again $\bar{l}$ and $\bar{\mu}$ are constants dependent on the properties of the plate only, and $\tilde{T}_o^*$ is a constant which depends also on T; $\tilde{T}_e^*$ is zero because T_e^* is an even function of z. For a homogeneous plate, $\bar{l} = l$, $\bar{\mu} = \mu$ and $\tilde{T}_o^* = 2\mu\gamma\tilde{T}_o$.

Using (31) in (17) shows that the moment equilibrium equation now reduces to the biharmonic equation

$$\nabla^4 \bar{w} = 0, \tag{33}$$

again the same thin-plate equation as for isothermal elasticity.

Hence in the thin-plate approximation, the temperature field occurs only in the boundary conditions, through (25) and (31). The governing two-dimensional field equations (27) and (33) are all as for the isothermal theory.

4. SYMMETRIC TEMPERATURE FIELD; STRETCHING MODE

When a symmetric temperature field $T_e(z)$ is imposed, its effect is obviously to induce a stretching deformation (Love, 1927) in which the in-plane displacements u, v are even functions of z whilst the transverse displacement w is an odd function of z. It is also reasonable to expect that the solution will reduce to that obtained by Spencer (1989) when the temperature field is zero. Moreover, the absence of constraint in the z-direction implies that expansion must take place in that direction. Accordingly we look for a solution of a form which is very similar to that obtained by Spencer, which was itself motivated by the "stretching" solutions of Michell (1900) for a homogeneous plate and by Kaprielian *et al.* (1988) for a laminate.

We propose a solution of the form

$$\begin{aligned} u &= \hat{u}(x, y) + F(z)\hat{\Delta}_{,x}, \\ v &= \hat{v}(x, y) + F(z)\hat{\Delta}_{,y}, \\ w &= G(z)\hat{\Delta} + K(z), \end{aligned} \tag{34}$$

where F is an even function of z and G and K are odd functions of z that need to be determined. In (34), $\hat{u}$, $\hat{v}$ and $\hat{\Delta}$ are functions of x and y which satisfy the equations

$$\hat{l}\hat{\Delta}_{,x} - \hat{\mu}\hat{\Omega}_{,y} = 0, \quad \hat{l}\hat{\Delta}_{,y} + \hat{\mu}\hat{\Omega}_{,x} = 0, \tag{35}$$

with

$$\hat{\Delta} = \hat{u}_{,x} + \hat{v}_{,y}, \quad \hat{\Omega} = \hat{v}_{,x} - \hat{u}_{,y}, \tag{36}$$

and hence, from (35),

$$\nabla^2\hat{\Delta} = 0, \quad \nabla^2\hat{\Omega} = 0. \tag{37}$$

From (27)–(29) we can therefore identify $\hat{u}$ and $\hat{v}$ as the corresponding thin-plate solution for an *equivalent* homogeneous plate with elastic constants $\hat{\lambda}$ and $\hat{\mu}$.

The stress corresponding to (34) is

$$\begin{aligned} \sigma_{xx} &= 2\mu\hat{u}_{,x} + \lambda(1+G')\hat{\Delta} + 2\mu F\hat{\Delta}_{,xx} + \lambda K' - \beta T_e, \\ \sigma_{yy} &= 2\mu\hat{v}_{,y} + \lambda(1+G')\hat{\Delta} + 2\mu F\hat{\Delta}_{,yy} + \lambda K' - \beta T_e, \\ \sigma_{xy} &= \mu(\hat{u}_{,y} + \hat{v}_{,x}) + 2\mu F\hat{\Delta}_{,xy}, \end{aligned} \tag{38}$$

and

$$\sigma_{xz} = \mu(F'+G)\hat{\Delta}_{,x},$$
$$\sigma_{yz} = \mu(F'+G)\hat{\Delta}_{,y},$$
$$\sigma_{zz} = \{\lambda+(\lambda+2\mu)G'\}\hat{\Delta}+(\lambda+2\mu)K'-\beta T_e, \tag{39}$$

where use has been made of (37), and primes denote differentiation with respect to z.

From $(39)_{1,2}$ we see that

$$\sigma_{xz,x}+\sigma_{yz,y} = \mu(F'+G)\nabla^2\hat{\Delta} = 0$$

so that the equilibrium equation $(10)_3$ immediately gives, with (11),

$$\sigma_{zz} = 0 \tag{40}$$

throughout. Thus $(39)_3$ requires G and K to be such that

$$G'(z) = -\lambda/(\lambda+2\mu) = -\eta \tag{41}$$

and

$$K' = \beta T_e/(\lambda+2\mu) = \gamma T_e. \tag{42}$$

Hence

$$G = -\int_0^z \eta(s)\,\mathrm{d}s, \quad K = \int_0^z \gamma(s)T_e(s)\,\mathrm{d}s. \tag{43}$$

The remaining equilibrium equations $(10)_{1,2}$, together with (38) and (39), require that

$$\{\mu(F'+G)\}' = (\mu\hat{l}/\hat{\mu})-l. \tag{44}$$

Furthermore, the properties of F and G require that $F'+G$ be an odd function of z, so that its value at $z = 0$ is zero. Hence (44) can be integrated to give

$$\mu(F'+G) = \frac{\hat{l}}{\hat{\mu}}M_0(z)-L_0(z), \tag{45}$$

where

$$L_0(z) = \int_0^z l(s)\,\mathrm{d}s, \quad M_0(z) = \int_0^z \mu(s)\,\mathrm{d}s. \tag{46}$$

It follows from $(39)_{1,2}$ that

$$\begin{pmatrix}\sigma_{xz}\\ \sigma_{yz}\end{pmatrix} = \left\{\frac{\hat{l}}{\hat{\mu}}M_0(z)-L_0(z)\right\}\begin{pmatrix}\hat{\Delta}_{,x}\\ \hat{\Delta}_{,y}\end{pmatrix} \tag{47}$$

and hence the shear traction τ on the planes z = constant is

$$\tau = \{(\hat{l}/\hat{\mu})M_0(z) - L_0(z)\} \text{ grad } \hat{\Delta}. \tag{48}$$

It is straightforward to confirm that τ is zero on $z = h$, thus satisfying condition (11). A simple integration of (45) then yields

$$F(z) = \int_0^z \left[-G(s) + \frac{1}{\mu(s)} \{(\hat{l}/\hat{\mu})M_0(s) - L_0(s)\} \right] \mathrm{d}s + k_0, \tag{49}$$

where we have used $(43)_1$ and k_0 is a constant.

The remaining stress components are now determined by

$$\begin{aligned} \sigma_{xx} &= l\hat{u}_{,x} + (l-2\mu)\hat{v}_{,y} + 2\mu F\hat{\Delta}_{,xx} - 2\mu\gamma T_e, \\ \sigma_{yy} &= (l-2\mu)\hat{u}_{,x} + l\hat{v}_{,y} + 2\mu F\hat{\Delta}_{,yy} - 2\mu\gamma T_e, \\ \sigma_{xy} &= \mu(\hat{u}_{,x} + \hat{v}_{,y}) + 2\mu F\hat{\Delta}_{,xy}, \end{aligned} \tag{50}$$

with $F(z)$ as given in (49).

The solution is completed when the disposable constant k_0 is determined. To assign this constant we may impose one further condition. If we wish the mean in-plane displacement components $\bar{u}$ and $\bar{v}$ to coincide with the mean displacements $\hat{u}$ and $\hat{v}$ for the equivalent plate, then k_0 must be such that

$$\int_0^h F(z)\ \mathrm{d}z = 0. \tag{51}$$

If alternatively it is required that the stress resultants N_{xx}, N_{yy} and N_{xy} coincide with those associated with the thin-plate solution for the equivalent plate, then integration of (50) and comparison with (25) show that then k_0 must be such that

$$\int_0^h \mu(z)F(z)\ \mathrm{d}z = 0. \tag{52}$$

It seems natural to choose (51) if in-plane displacements are prescribed on the edge of the plate, and to choose (52) if traction boundary conditions are prescribed there.

In many respects the solution is remarkably simple, and very similar to that obtained by Spencer (1989). Apart from its effect on the edge conditions applying to the approximate thin-plate equations, the temperature field introduces one additional term in the displacement field, namely the function $K(z)$ in the transverse displacement w, and the term $-\beta^* T_e$ in the in-plane stress components σ_{xx} and σ_{yy}. Otherwise the solution is the same as that obtained for the isothermal case.

5. ANTI-SYMMETRIC TEMPERATURE FIELD; BENDING MODE

Just as the symmetric part T_e of the temperature field produces stretching but no bending of the middle surface, so the anti-symmetric part T_o produces bending but no stretching. We are again motivated by previous bending solutions (Michell, 1900; Kaprielian *et al.*, 1988) and their generalization to inhomogeneous plates by Rogers (1989), together with the experience of the previous section, Section 4. Accordingly we now look for solutions of the form

$$\begin{aligned} u &= \{A(z)\tilde{w} + B(z)\nabla^2\tilde{w}\}_{,x}, \\ v &= \{A(z)\tilde{w} + B(z)\nabla^2\tilde{w}\}_{,y}, \\ w &= \tilde{w} + C(z)\nabla^2\tilde{w} + D(z). \end{aligned} \tag{53}$$

This is the same form as that of Rogers (1989) with the sole exception of the additional term $D(z)$ in the transverse displacement w. The function $\tilde{w}(x, y)$ is chosen to be the transverse displacement of the equivalent homogeneous plate with elastic constants $\tilde{\lambda}$ and $\tilde{\mu}$ and so satisfies the two-dimensional biharmonic equation

$$\nabla^4 \tilde{w} = 0. \tag{54}$$

The functions A, B, C and D must be determined such that equilibrium is maintained and the boundary conditions (11) and edge conditions are satisfied.

The stress corresponding to (53) is

$$\begin{aligned} \sigma_{xx} &= \lambda(A+C')\nabla^2\tilde{w}+2\mu(A\tilde{w}+B\nabla^2\tilde{w})_{,xx}+\lambda D'-\beta T_o, \\ \sigma_{yy} &= \lambda(A+C')\nabla^2\tilde{w}+2\mu(A\tilde{w}+B\nabla^2\tilde{w})_{,yy}+\lambda D'-\beta T_o, \\ \sigma_{xy} &= 2\mu(A\tilde{w}+B\nabla^2\tilde{w})_{,xy}, \end{aligned} \tag{55}$$

and

$$\begin{aligned} \sigma_{xz} &= \mu\{(A'+1)\tilde{w}+(B'+C)\nabla^2\tilde{w}\}_{,x}, \\ \sigma_{yz} &= \mu\{(A'+1)\tilde{w}+(B'+C)\nabla^2\tilde{w}\}_{,y}, \\ \sigma_{zz} &= \{\lambda A+(\lambda+2\mu)C'\}\nabla^2\tilde{w}+(\lambda+2\mu)D'-\beta T_o, \end{aligned} \tag{56}$$

where use has been made of (54).

Once again, as in Section 4, we start with equilibrium equation $(10)_3$; substitution from (56) yields the relation

$$[\mu(A'+1)+\{\lambda A+(\lambda+2\mu)C'\}']\nabla^2\tilde{w}+\{(\lambda+2\mu)D'-\beta T_o\}' = 0. \tag{57}$$

Hence, A, C and D must satisfy

$$\mu(A'+1)+\{\lambda A+(\lambda+2\mu)C'\}' = 0 \tag{58}$$

and

$$\{(\lambda+2\mu)D'-\beta T_o\}' = 0. \tag{59}$$

Similarly the remaining equilibrium equations imply that A, B and C must satisfy

$$\{\mu(A'+1)\}' = 0 \tag{60}$$

and

$$2\mu A+\lambda(A+C')+\{\mu(B'+C)\}' = 0. \tag{61}$$

Zero traction on the lateral surfaces requires the conditions that

$$A'+1 = 0, \quad B'+C = 0, \quad \lambda A+(\lambda+2\mu)C' = 0, \quad (\lambda+2\mu)D'-\beta T_o = 0 \quad \text{on} \quad z = \pm h. \tag{62}$$

Hence (58)–(60), show that, throughout the plate,

$$A'+1 = 0, \quad \lambda A+(\lambda+2\mu)C' = 0 \tag{63}$$

and

$$(\lambda+2\mu)D'-\beta T_o = 0. \tag{64}$$

Finally, symmetry about $z = 0$ implies that both $A(z)$ and $B(z)$ are zero on the middle surface.

The equations and conditions governing A, B and C are exactly the same as in the solution derived by Rogers (1989), giving

$$\begin{aligned} A &= -z, \\ B &= -\int_0^z sl(s)R_0(s)\,\mathrm{d}s-\{L_1(h)-L_1(z)\}R_0(z)+N_2(z)-Cz, \\ C &= N_1(z)+k_1. \end{aligned} \tag{65}$$

Here k_1 is a constant, and R_0, L_1, N_1 and N_2 are integrals involving only the elastic properties:

$$R_0(z) = \int_0^z \frac{1}{\mu(s)}\,\mathrm{d}s, \quad L_1(z) = \int_0^z sl(s)\,\mathrm{d}s, \tag{66}$$

$$N_1(z) = \int_0^z s\eta(s)\,\mathrm{d}s, \quad N_2(z) = \int_0^z s^2\eta(s)\,\mathrm{d}s, \tag{67}$$

with the suffices 0, 1, 2 denoting the power of s in the integrand. Also (56), (63) and (64) show that, as in Rogers (1989),

$$\sigma_{zz} = 0 \tag{68}$$

throughout, and the shear traction $\boldsymbol{\tau}$ on surfaces z = constant is given by

$$\boldsymbol{\tau} = \{L_1(z)-L_1(h)\}\ \mathrm{grad}\ \nabla^2\tilde{w}. \tag{69}$$

The only new feature is the term $D(z)$ given by (64) as

$$D = \int_0^z \gamma T_o\,\mathrm{d}s+k_2, \tag{70}$$

where γ is defined in (8) and k_2 is another disposable constant.

The remaining stress components can now be written as

$$\begin{aligned} \sigma_{xx} &= -z\{l\tilde{w}_{,xx}+(l-2\mu)\tilde{w}_{,yy}\}+2\mu B\nabla^2\tilde{w}_{,xx}-\beta^* T_o, \\ \sigma_{yy} &= -z\{(l-2\mu)\tilde{w}_{,xx}+l\tilde{w}_{,yy}\}+2\mu B\nabla^2\tilde{w}_{,yy}-\beta^* T_o, \\ \sigma_{xy} &= 2\mu(-z\tilde{w}+B\nabla^2\tilde{w})_{,xy}, \end{aligned} \tag{71}$$

where B is given in (65).

The two disposable constants k_1 and k_2, like k_0 in Section 4, may be assigned any convenient values. If edge deflections are specified then a suitable choice would be to make $w = \tilde{w}$ at $z = 0$ so that the middle surface deflection coincides with that of the equivalent plate. Then $C(0) = D(0) = 0$, with

$$k_1 = k_2 = 0. \tag{72}$$

Alternatively, if edge bending moments are specified then a more suitable choice would be to make the bending moments coincide with those in the equivalent plate. In this case, (15), (71) and (31) show that the relevant condition is

$$\int_0^h z\mu(z)B(z)\,\mathrm{d}z = 0. \tag{73}$$

This yields a value for k_1, but leaves k_2 arbitrary, being equivalent only to a rigid translation in the z-direction.

Once again we note the straightforward derivation of the solution, and its similarity to that obtained by Rogers (1989) for isothermal problems. Again, apart from its effect on the edge conditions imposed on the thin-plate equations, the temperature field introduces one additional term $D(z)$ in the transverse displacement w and the term $-\beta^* T_0$ in σ_{xx} and σ_{yy}. Indeed, we observe that when the two solutions of Section 4 and the above are superposed, the overall effect is to superpose the "thermal displacement" vector

$$\left(0, 0, \int_0^z \frac{\beta(s)}{\lambda(s)+2\mu(s)}\,T(s)\,\mathrm{d}s\right) \tag{74}$$

onto the displacement field derived by Spencer (1989) and Rogers (1989), together with an additional term $-\beta^* T$ onto the stress components σ_{xx} and σ_{yy}.

6. LAMINATED PLATES

A very important special case of an inhomogeneous plate is that of a laminate, consisting of a number of uniform layers of possibly different thicknesses and of different homogeneous elastic materials. This was the case considered by Kaprielian *et al.* (1988) for plane stress stretching and bending, and also considered as a special case by both Spencer (1989) and Rogers (1989).

The laminate theories in these three papers are equivalent but were derived by different methods. The theory in the first is based on a generalization of Michell's solution for a single layer, such that each layer solution contains a number of disposable constants and does not require the shear tractions on the lateral surface to vanish. It was found that these arbitrary constants were sufficient to satisfy all the interface traction and displacement continuity conditions as well as the vanishing traction conditions (11) on the lateral surfaces. The continuity conditions naturally led to a system of recurrence relations for the arbitrary constants, which could be solved in a straightforward manner to produce the required exact solution. In the other two papers it was shown how these relations could be recovered from the inhomogeneous plate theory.

In this section we show an equivalent but different formulation in which the solution is expressed in terms of constants associated with each interlaminar boundary rather than with each layer. These constants are also determind by recurrence formulae in a straightforward manner, and can be interpreted directly in terms of interlaminar displacement, shear tractions, etc.

In Kaprielian *et al.* (1988), the middle surface of each layer therefore had a significance which no longer applies in the present inhomogeneous context; the significant positions in a laminate are the interfaces at which adjacent layers are perfectly bonded. Accordingly some of our notation differs from that of these previous three papers.

We consider a symmetric laminated plate (refer to Fig. 1) comprised of $2N+1$ homogeneous laminae, with the ith laminae above and below the mid-plane $z = 0$ being identical in material and thicknesses. As before, any quantity related to the ith layer is identified by the index i, and the layers are consecutively numbered from $i = 0$, the layer containing the mid-plane, to $i = N$, the layers adjacent to the lateral surfaces. The ith layers have uniform thickness $2h_i$ and Lamé elastic constants λ_i, μ_i and stress–temperature coefficient β_i, with

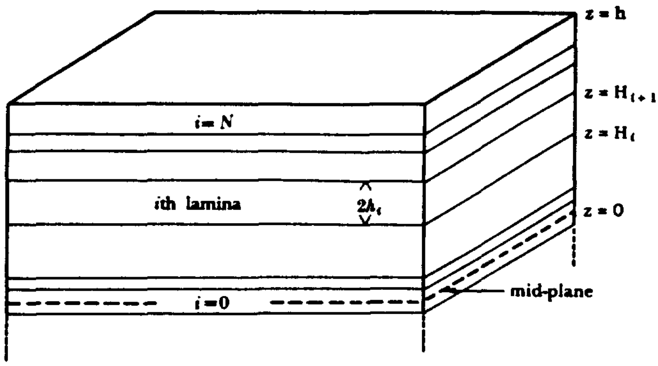

Fig. 1. Laminate geometry.

associated constants l_i, η_i, λ_i^*, γ_i and β_i^*. The overall thickness is $2h$ with

$$h = h_0 + 2\sum_{i=1}^{N} h_i. \tag{75}$$

We also denote by H_i the distance from the mid-plane of the plate $z = 0$ to the *interface* $z = H_i$ between the $(i-1)$th and ith layers, unlike the previous three papers where H_i denotes the distance from $z = 0$ to the middle surface of the ith layer. Hence the ith layer is defined by

$$H_i \leqslant z = z_i \leqslant H_{i+1} \quad (i = 1, 2, \ldots, N) \tag{76}$$

where z_i denotes the z variable in the ith layer, and we have

$$H_1 = h_0, \quad H_{N+1} = h, \quad H_i = h_0 + 2\sum_{r=1}^{i-1} h_r \quad (i = 1, 2, \ldots, N) \tag{77}$$

with, for example, from (46),

$$M_0(z_0) = \mu_0 z_0, \quad M_0(z_1) = \mu_0 h_0 + \mu_1(z_1 - H_1),$$
$$M_0(z_i) = \mu_0 h_0 + 2\sum_{r=1}^{i-1} \mu_r h_r + \mu_i(z_i - H_i), \quad i > 1. \tag{78}$$

An important additional notation is to introduce the *interlaminar constants* $M_0^{(i)}$, $L_0^{(i)}$, $L_1^{(i)}$, $N_1^{(i)}$ and $N_2^{(i)}$, where $M_0^{(i)} = M_0(H_i)$, $L_0^{(i)} = L_0(H_i)$, $L_1^{(i)} = L_1(H_i)$, $N_1^{(i)} = N_1(H_i)$ and $N_2^{(i)} = N_2(H_i)$. Hence, from (46),

$$M_0^{(i)} = \mu_0 h_0 + 2\sum_{r=1}^{i-1} \mu_r h_r, \quad L_0^{(i)} = l_0 h_0 + 2\sum_{r=1}^{i-1} l_r h_r. \tag{79}$$

Then, for $i > 0$,

$$M_0(z_i) = M_0^{(i)} + \mu_i(z_i - H_i), \quad L_0(z_i) = L_0^{(i)} + l_i(z_i - H_i), \tag{80}$$

with

$$M_0^{(i+1)} = M_0^{(i)} + 2\mu_i h_i, \quad L_0^{(i+1)} = L_0^{(i)} + 2l_i h_i. \tag{81}$$

In a similar fashion and notation we obtain, from (66) and (67),

$$L_1(z_0) = \tfrac{1}{2} l_0 z_0^2, \quad L_1(z_i) = L_1^{(i)} + \tfrac{1}{2} l_i (z_i^2 - H_i^2),$$
$$N_1(z_0) = \tfrac{1}{2}\eta_0 z_0^2, \quad N_1(z_i) = N_1^{(i)} + \tfrac{1}{2}\eta_i(z_i^2 - H_i^2),$$
$$N_2(z_0) = \tfrac{1}{3}\eta_0 z_0^3, \quad N_2(z_i) = N_2^{(i)} + \tfrac{1}{3}\eta_i(z_i^3 - H_i^3), \tag{82}$$

so that

$$L_1^{(0)} = \tfrac{1}{2}l_0h_0^2, \quad L_1^{(i+1)} = L_1^{(i)} + \tfrac{1}{2}l_i(H_{i+1}^2 - H_i^2), \tag{83}$$

$$N_1^{(0)} = \tfrac{1}{2}\eta_0h_0^2, \quad N_1^{(i+1)} = N_1^{(i)} + \tfrac{1}{2}\eta_i(H_{i+1}^2 - H_i^2), \tag{84}$$

$$N_2^{(0)} = \tfrac{1}{3}\eta_0h_0^3, \quad N_2^{(i+1)} = N_2^{(i)} + \tfrac{1}{3}\eta_i(H_{i+1}^3 - H_i^3). \tag{85}$$

Then in particular we have

$$\hat{\mu} = \frac{1}{h}M_0^{(N+1)}, \quad \hat{l} = \frac{1}{h}L_0^{(N+1)}, \quad \tilde{\mu} = \frac{3}{2h^3}M_2^{(N+1)}, \quad \tilde{l} = \frac{3}{2h^3}L_2^{(N+1)}. \tag{86}$$

The integrals involving the temperature field cannot be expressed in such a simple fashion, and involve quadratures of the temperature functions T_e and T_o. From (43) and (70),

$$K(z_0) = \gamma_0\int_0^{z_0} T_e(s)\ \mathrm{d}s, \quad K(z_i) = K^{(i)} + \gamma_i\int_{H_i}^{z_i} T_e(s)\ \mathrm{d}s, \tag{87}$$

$$D(z_0) = \gamma_0\int_0^{z_0} T_o(s)\ \mathrm{d}s, \quad D(z_i) = D^{(i)} + \gamma_i\int_{H_i}^{z_i} T_o(s)\ \mathrm{d}s, \tag{88}$$

where $K^{(i)} = K(H_i)$, $D^{(i)} = D(H_i)$ and, without loss of generality, we choose $k_2 = 0$.

The complete solutions for both stretching and bending can be formulated in terms of the interlaminar constants. Thus, in the stretching mode, $K(z_i)$ is given in (87) and

$$G(z_0) = -\eta_0z_0, \quad G(z_i) = G^{(i)} - \eta_i(z_i - H_i), \quad i = 1,\dots,N-1, \tag{89}$$

where $G^{(i)} = G(H_i)$. Therefore

$$G^{(0)} = -\eta_0h_0, \quad G^{(i+1)} = G^{(i)} - 2\eta_ih_i. \tag{90}$$

The function $F(z)$ given in (49) is much more complicated. However, it is easy to show that

$$F(z_0) = (\hat{\eta} - \tfrac{1}{2}\eta_0)z_0^2 + k_0, \quad 2\hat{\eta} + 1 = \hat{l}/\hat{\mu} \tag{91}$$

so that, denoting $F^{(i)} = F(H_i)$,

$$F^{(0)} = (\hat{\eta} - \tfrac{1}{2}\eta_0)h_0^2 + k_0, \tag{92}$$

and for $i = 1, 2, \dots, N$ we obtain

$$F(z_i) = F^{(i)} + \frac{1}{\mu_i}S^{(i)}(z_i - H_i) + (\hat{\eta} - \tfrac{1}{2}\eta_i)(z_i - H_i)^2, \tag{93}$$

where

$$S^{(i)} = 2(\hat{\eta} + 1)M_0^{(i)} - L_0^{(i)}. \tag{94}$$

Hence

$$F^{(i+1)} = F^{(i)} + 2S^{(i)}h_i/\mu_i + 2(2\hat{\eta} - \eta_i)h_i^2. \tag{95}$$

The method for determining the "stretching" constants is then a straightforward recurrence procedure. We first determine $\hat{l}$ and $\hat{\mu}$ to give $\hat{\eta}$, and then use (81), (87), (90), (92), (94) and (95) recursively to give $M_0^{(i)}$, $L_0^{(i)}$, $K^{(i)}$, $G^{(i)}$, $S^{(i)}$ and $F^{(i)}$ for $i = 0, 1, \ldots, N$.

The value of k_0 can be taken as zero for this computation; this effectively means that one first computes not $F^{(i)}$ but $\bar{F}^{(i)}$, where

$$\bar{F}^{(i)} = F^{(i)} - k_0. \tag{96}$$

One can then use these values for $\bar{F}^{(i)}$ in the relevant condition, such as (51) or (52), to give k_0. Thus, for (51), we write

$$0 = \int_0^h \{\bar{F}(z) + k_0\}\, \mathrm{d}z$$

so that

$$\begin{aligned} k_0 &= -\frac{1}{h}\int_0^h \bar{F}(z)\, \mathrm{d}z \\ &= -\frac{1}{h}\left[\tfrac{1}{2}h_0\bar{F}^{(0)} + 2\sum_{i=0}^{N-1}\{h_i\bar{F}^{(i)} + h_i^2 S_i/\mu_i + \tfrac{4}{3}h_i^3(2\hat{\eta} - \eta_i)\}\right]. \end{aligned} \tag{97}$$

If condition (52) is to be used, then k_0 is given by

$$\begin{aligned} k_0 &= -\frac{1}{h\hat{\mu}}\int_0^h \mu(z)\bar{F}(z)\, \mathrm{d}z \\ &= -\frac{1}{h\hat{\mu}}\left[\tfrac{1}{2}h_0\mu_0\bar{F}^{(0)} + 2\sum_{i=0}^{N-1}\{h_i\mu_i\bar{F}^{(i)} + h_i^2 S_i + \tfrac{4}{3}h_i^3\mu_i(2\hat{\eta} - \eta_i)\}\right]. \end{aligned} \tag{98}$$

The solution for bending may be treated similarly. The temperature term $D(z_i)$ is given in (88). Denoting $C^{(i)} = C(H_i)$ we have, from (65) and (82),

$$C(z_0) = \tfrac{1}{2}\eta_0 z_0^2 + k_1, \quad C(z_i) = C^{(i)} + \tfrac{1}{2}\eta_i(z_i^2 - H_i^2), \tag{99}$$

so that

$$C^{(0)} = \tfrac{1}{2}\eta_0 h_0^2 + k_1, \quad C^{(i+1)} = C^{(i)} + \tfrac{1}{2}\eta_i(H_{i+1}^2 - H_i^2). \tag{100}$$

The more complicated function $B(z)$ takes the form, from (65),

$$B(z_0) = \tfrac{1}{6}(\eta_0 + 2)z_0^3 - (C^{(0)} - \tfrac{1}{2}\eta_0 h_0^2 - L_1^{(N+1)}/\mu_0)z_0 \tag{101}$$

in the central layer, and in the ith layer ($i > 0$) we have

$$B(z_i) = B^{(i)} + \tfrac{1}{6}(\eta_i + 2)(z_i + 2H_i)(z_i - H_i)^2 - \{C^{(i)} + (L_1^{(N+1)} - L_1^{(i)})/\mu_i\}(z_i - H_i), \tag{102}$$

where $B^{(i)} = B(H_i)$. Hence

$$B^{(0)} = \tfrac{1}{3}(2\eta_0 + 1)h_0^3 - C^{(0)}h_0 - L_1^{(N+1)}/\mu_0 \tag{103}$$

and

$$B^{(i+1)} = B^{(i)} - 2h_i\{C^{(i)} + (L_1^{(N+1)} - L_1^{(i)})/\mu_i\} + \tfrac{2}{3}h_i^2(3H_i + 2h_i)(\eta_i + 2). \tag{104}$$

The procedure for determining the "bending" constants is as straightforward as for the stretching case. We first determine $L_1^{(N+1)} - L_1^{(i)}$ successively for i decreasing from N to 1 using the relation [from (83)]

$$L_1^{(N+1)} - L_1^{(i)} = L_1^{(N+1)} - L_1^{(i+1)} + \tfrac{1}{2}l_i(H_{i+1}^2 - H_i^2). \tag{105}$$

From (69) we see that these values effectively give the interlaminar shear tractions $\tau^{(i)}$. We then use (88), (100), (103) and (104) recursively to give $D^{(i)}$, $C^{(i)}$ and $B^{(i)}$ for $i = 0, 1, \ldots, N$.

The value for k_1 can be treated in the same way as for k_0. We first carry out the above computations with $k_1 = 0$. If (72) is the appropriate condition then obviously this completes the solution; otherwise we have effectively computed not $C^{(i)}$ and $B^{(i)}$ but $\bar{C}^{(i)}$ and $\bar{B}^{(i)}$ with

$$\bar{C}(z) = C(z) - k_1, \quad \bar{B}(z) = B(z) + k_1 z$$

and

$$\bar{C}^{(i)} = C^{(i)} - k_1, \quad \bar{B}^{(i)} = B^{(i)} + k_1 H_i. \tag{106}$$

These values can then be used in condition (72), say, to give k_1, since then

$$0 = \int_0^h z\mu(z)\{\bar{B}(z) - k_1 z\}\,\mathrm{d}z$$

so that

$$\begin{aligned} k_1 &= \frac{3}{\bar{\mu}h^3}\int_0^h z\mu(z)\bar{B}(z)\,\mathrm{d}z \\ &= \frac{1}{\bar{\mu}h^3}\Bigg[\{\tfrac{1}{10}(\eta_0+2)\mu_0 z_0^2 + L_1^{(N+1)} - (C^{(0)} - \tfrac{1}{2}\mu_0 h_0^2)\mu_0\} z_0^3 \\ &\quad + 6\sum_{i=1}^{N} h_i^2\{\mu_i(\bar{B}^{(i)} - C^{(i)}) - L_1^{(N+1)} + L_1^{(i)} + \tfrac{1}{3}h_i\mu_i(\eta_i+2)(2H_i+h_i)\}\Bigg]. \end{aligned} \tag{107}$$

When the interlaminar constants have been determined, then straightforward substitution into expressions (93), (89) and (87) for F, G and K, and then into the relevant formulae in Section 4, gives exact three-dimensional solutions for thermal stretching of the laminated plate. Exact solutions for thermal bending are obtained by using (101), (102), (100) and (88) for B, C and D and then substituting into the relevant formulae in Section 5. In both cases, the solutions are given for any plate problem, under edge loading, provided the relevant solution to the thin-plate equations for the equivalent plate is known.

In these solutions, it is seen from (40) and (68) that the transverse normal stress component σ_{zz} is identically zero throughout the plate, whether it is a laminate or more generally inhomogeneous. In general, the shear traction τ vanishes only at the external surfaces; its interlaminar values are given by (48), (69), (79) and (83).

Acknowledgement—This work was carried out with the support of NATO Collaborative Research Grant 0496/87; this support is gratefully acknowledged.

REFERENCES

Christensen, R. M. (1979). *Mechanics of Composite Materials*. John Wiley, New York.

Gregory, R. D. and Wan, F. Y. M. (1984). Decaying states of plane strain in a semi-infinite strip and boundary conditions for plate theory. *J. Elasticity* **14**, 27–64.

Gregory, R. D. and Wan, F. Y. M. (1985). On plate theories and Saint-Venant's principle. *Int. J. Solid Structures* **21**, 1005–1024.

Kaprielian, P. V. (1985). The effects of holes and free edges on the stress in laminated plates. Chapter 5, Ph.D. Thesis, University of Nottingham.

Kaprielian, P. V., Rogers, T. G. and Spencer, A. J. M. (1988). Theory of laminated elastic plates, I. Isotropic laminae. *Phil. Trans. R. Soc. Lond.* **A324**, 565–594.

Love, A. E. H. (1927). *The Mathematical Theory of Elasticity*, 4th Edn. Cambridge University Press, Cambridge.

Michell, J. H. (1900). On the direct determination of stress in an elastic solid, with applications to the theory of plates. *Proc. Lond. Math. Soc.* **31**, 100–124.

Rogers, T. G. (1989). Exact three-dimensional bending solutions for inhomogeneous and laminated elastic plates. In *Elasticity, Mathematical Methods and Applications* (the Ian Sneddon 70th Birthday Volume) (Edited by G. Eason and R. W. Ogden). Ellis Horwood Limited, Chichester.

Spencer, A. J. M. (1989). Three-dimensional elasticity solutions for stretching of inhomogeneous and laminated plates. In *Elasticity, Mathematical Methods and Applications* (the Ian Sneddon 70th Birthday Volume) (Edited by G. Eason and R. W. Ogden). Ellis Horwood Limited, Chichester.

Timoshenko, S. P. and Woinowsky-Krieger, S. (1959). *Theory of Plates and Shells*, 2nd Edn. McGraw-Hill, New York.